

The Nevanlinna parametrization for q -Lommel polynomials in the indeterminate case

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Abstract

The Hamburger moment problem for the q -Lommel polynomials which are related to the Hahn-Exton q -Bessel function is known to be indeterminate for a certain range of parameters. In this paper, the Nevanlinna parametrization for the indeterminate case is provided in an explicit form. This makes it possible to describe all N-extremal measures of orthogonality. Moreover, a linear and quadratic recurrence relation are derived for the moment sequence, and the asymptotic behavior of the moments for large powers is obtained with the aid of appropriate estimates.

Keywords: q -Lommel polynomials, Nevanlinna parametrization, measure of orthogonality, moment sequence

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1 Introduction

The Lommel polynomials represent a class of orthogonal polynomials known from the theory of Bessel functions. Several q -analogues of the Lommel polynomials have been introduced and studied in [16, 15, 17]. One of the three commonly used q -analogues of the Bessel function of the first kind is known as the Hahn-Exton q -Bessel function (sometimes also called the third Jackson q -Bessel function or ${}_1\phi_1$ q -Bessel function). It is defined by

$$J_\nu(z; q) = z^\nu \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; q^{\nu+1}; q, qz^2). \quad (1)$$

It is of importance that $J_\nu(z; q)$ satisfies the recurrence relation

$$J_{\nu+1}(z; q) - \left(z + \frac{1 - q^\nu}{z} \right) J_\nu(z; q) + J_{\nu-1}(z; q) = 0.$$

By iterating this rule one arrives at the formula

$$J_{\nu+n}(z; q) = h_{n,\nu}(z^{-1}; q) J_\nu(z; q) - h_{n-1,\nu+1}(z^{-1}; q) J_{\nu-1}(z; q) \quad (2)$$

where $h_{m,\nu}(w; q)$ are polynomials in q^ν and Laurent polynomials in w , see [15] for more details. This is a familiar situation, with equation (2) being analogous to the well known relation between the Lommel polynomials and the Bessel functions, cf. [26, Chapter 9]. Thus the polynomials $h_{m,\nu}(w; q)$ can be referred to as the q -Lommel polynomials.

On one hand, the polynomials $h_{n,\nu}(w; q)$ can be treated as orthogonal Laurent polynomials in the variable w . The corresponding orthogonality relation has been described in [15]. On the other hand, $h_{n,\nu}(w; q)$ are also orthogonal polynomials in the variable q^ν . In Theorem 3.6 and Corollary 3.7 in [17], Koelink described a corresponding measure of orthogonality. It turns out that the measure of orthogonality is supported on the zeros of the Hahn-Exton q -Bessel function considered as a function of the order ν . Moreover, the measure of orthogonality is unique if $w^{-2} \leq q$ or $w^{-2} \geq q^{-1}$. For $q < w^{-2} < q^{-1}$, however, the corresponding Hamburger moment problem is indeterminate and so there exist infinitely many measures of orthogonality. The measure described in [17] represents a Nevanlinna (or N-) extremal solution of the indeterminate Hamburger moment problem, and it can be seen to correspond to the Friedrichs extension of the underlying Jacobi matrix operator.

Let us also remark that the q -Lommel polynomials admit another interpretation in the framework of a birth and death process with exponentially growing birth and death rates. More precisely, the birth rate is supposed to be $\lambda_n = w^{-2} q^{-n}$ while the death rate is $\mu_n = q^{-n}$ (or vice versa). See, for example, [13] for more information on the subject.

As already pointed out in [17], it is of interest and in fact a fundamental question to determine all possible measures of orthogonality in terms of the Nevanlinna parametrization. An explicit solution of this problem becomes the main goal of the current paper. To achieve it we heavily rely on the knowledge of the generating function for the q -Lommel polynomials. Having the Nevanlinna parametrization at hand it is straightforward to describe all N-extremal measures of orthogonality. The case when $w = 1$ turns out to be somewhat special and requires additional efforts though no new ideas are in principle needed. To our best knowledge, formulas for this particular case have been omitted in the past research works on the q -Lommel polynomials.

The measures of orthogonality we are going to describe are necessarily discrete. To reveal a bit their structure we have a closer look at the asymptotics of the mass points and the corresponding weights of such a measure. To this end, we make use of some known results concerned with the asymptotic behavior of the roots of the q -Bessel functions. A brief summary of basic facts and references on this subject is provided

in Appendix. Moreover, we were able to complete these facts with some additional details.

Furthermore, we pay some attention to the sequence of moments related to the q -Lommel polynomials. By Favard's theorem, the moments are uniquely determined by the coefficients in the recurrence relation for the q -Lommel polynomials and otherwise they are independent of a particular choice of the measure of orthogonality in the indeterminate case. It does not seem that the moment sequence can be found explicitly. We provide at least a linear and quadratic recurrence relation for it and describe qualitatively its asymptotic behavior for large powers.

Let us note that throughout the whole paper the parameter q is assumed to satisfy $0 < q < 1$. Furthermore, as far as the basic (or q -) hypergeometric series are concerned, as well as other q -symbols and functions, we follow the notation of Gasper and Rahman [12].

2 The Nevanlinna functions for q -Lommel polynomials

2.1 The q -Lommel polynomials

In the current paper we prefer to work directly with the ${}_1\phi_1$ basic hypergeometric function and do not insist on its interpretation as the q -Bessel function in accordance with (1). This leads us to using a somewhat modified notation if compared to that usually used in connection with q -Bessel functions, for instance, in [17]. Moreover, the notation used in this paper may stress some similarity of the Hamburger moment problem for the q -Lommel polynomials with the same problem for the Al-Salam-Carlitz II polynomials. The Hamburger moment problem is actually known to be indeterminate for particular values of parameters in both cases but there are also some substantial differences, see [5, Section 4].

Thus we write $a > 0$ instead of w^{-2} and $x \in \mathbb{C}$ instead of q^ν . The basic recurrence relation we are going to study, defining a sequence of monic orthogonal polynomials $\{F_n(a, q; x)\}_{n=0}^\infty$ (in the variable x and depending on two parameters a and q), reads

$$u_{n+1} = (x - (a + 1)q^{-n})u_n - aq^{-2n+1}u_{n-1}, \quad n \in \mathbb{Z}_+ \quad (3)$$

(\mathbb{Z}_+ standing for nonnegative integers). As usual, the initial conditions are imposed in the form $F_{-1}(a, q; x) = 0$ and $F_0(a, q; x) = 1$. In order to be able to compare some results derived below with the already known results on the q -Lommel polynomials let us remark that the q -Lommel polynomials $h_{n,\nu}(w; q)$ introduced in (2) are related to the monic polynomials $F_n(a, q; x)$ by the formula

$$h_{n,\nu}(w; q) = (-1)^n w^n q^{n(n-1)/2} F_n(w^{-2}, q; q^\nu).$$

From (3) one immediately deduces the symmetry property

$$a^n F_n(a^{-1}, q; x) = F_n(a, q; ax), \quad n \in \mathbb{Z}_+.$$

This suggests that one can restrict values of the parameter a to the interval $0 < a < 1$. We usually try, however, to formulate our results for both cases, $a < 1$ and $a > 1$, for the sake of completeness. The case $a = 1$ is somewhat special and should be treated separately.

Letting

$$G_n(a, q; x) = q^{1-n} F_{n-1}(a, q; qx), \quad n \in \mathbb{Z}_+, \quad (4)$$

we get a second linearly independent solution of (3), a sequence of monic polynomials $\{G_n(a, q; x)\}$ fulfilling the initial conditions $G_0(a, q; x) = 0$ and $G_1(a, q; x) = 1$. Normalizing the monic polynomials $F_n(a, q; x)$ we get an orthonormal polynomial sequence $\{P_n(a, q; x)\}_{n=0}^\infty$. Explicitly,

$$P_n(a, q; x) = a^{-n/2} q^{n^2/2} F_n(a, q; x), \quad n \in \mathbb{Z}_+. \quad (5)$$

The polynomials of the second kind, $Q_n(a, q; x)$, are related to the monic polynomials $G_n(a, q; x)$ by a similar equality,

$$Q_n(a, q; x) = a^{-n/2} q^{n^2/2} G_n(a, q; x), \quad n \in \mathbb{Z}_+, \quad (6)$$

and obey the initial conditions $Q_0(a, q; x) = 0$, $Q_1(a, q; x) = \sqrt{q/a}$.

Note that polynomials $P_n(a, q; x)$ solve the second-order difference equation

$$\sqrt{a} q^{-n+1/2} v_{n-1} + ((a+1)q^{-n} - x) v_n + \sqrt{a} q^{-n-1/2} v_{n+1} = 0, \quad n \in \mathbb{Z}_+,$$

with the initial conditions $P_{-1}(a, q; x) = 0$ and $P_0(a, q; x) = 1$. Denote by α_n and β_n the coefficients in this difference equation,

$$\alpha_n = a^{1/2} q^{-n-1/2}, \quad \beta_n = (a+1)q^{-n}, \quad n \in \mathbb{Z}_+. \quad (7)$$

The difference equation can be interpreted as the formal eigenvalue equation for the Jacobi matrix

$$J = J(a, q) = \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \quad (8)$$

Then $(P_0(x), P_1(x), P_2(x), \dots)$ is a formal eigenvector (where $P_j(x) \equiv P_j(a, q; x)$). Let us emphasize that J is positive on the subspace in $\ell^2(\mathbb{Z}_+)$ formed by sequences with only finitely many nonzero entries, i.e. on the linear hull of the canonical basis in $\ell^2(\mathbb{Z}_+)$. Actually, it is not difficult to verify that for every $N \in \mathbb{Z}_+$ and $\xi \in \mathbb{R}^{N+1}$,

$$\sum_{n=0}^N \beta_n \xi_n^2 + 2 \sum_{n=0}^{N-1} \alpha_n \xi_n \xi_{n+1} = a \xi_0^2 + q^{-N} \xi_N^2 + \sum_{n=0}^{N-1} q^{-n} \left(\left(\frac{a}{q} \right)^{1/2} \xi_{n+1} + \xi_n \right)^2 \geq 0. \quad (9)$$

Recurrence (3) can be solved explicitly in the particular case when $x = 0$. One finds that

$$F_n(a, q; 0) = (-1)^n q^{-n(n-1)/2} \frac{1 - a^{n+1}}{1 - a}, \quad G_n(a, q; 0) = (-1)^{n+1} q^{-n(n-1)/2} \frac{1 - a^n}{1 - a},$$

for $n \in \mathbb{Z}_+$ and $a \neq 1$. Consequently,

$$P_n(a, q; 0) = (-1)^n q^{n/2} a^{-n/2} \frac{1 - a^{n+1}}{1 - a}, \quad Q_n(a, q; 0) = (-1)^{n+1} q^{n/2} a^{-n/2} \frac{1 - a^n}{1 - a}. \quad (10)$$

The quantities $P_n(1, q; 0)$ and $Q_n(1, q; 0)$ can be obtained from (10) in the limit $a \rightarrow 1$,

$$P_n(1, q; 0) = (-1)^n q^{n/2} (n+1), \quad Q_n(1, q; 0) = (-1)^{n+1} q^{n/2} n.$$

2.2 The generating function

A formula for the generating function for the q -Lommel polynomials has been derived in [16, Eq. (4.22)]. Here we reproduce the formula and provide its proof since it is quite crucial for the computations to follow of the Nevanlinna functions A , B , C , and D .

Proposition 1. *Let $a > 0$. The generating function for the polynomials $F_n(a, q; x)$ equals*

$$\sum_{n=0}^{\infty} q^{n(n-1)/2} F_n(a, q; x) (-t)^n = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} (-xt)^k}{(t; q)_{k+1} (at; q)_{k+1}} = \frac{{}_2\phi_2(q, 0; qt, qat; q, xt)}{(1-t)(1-at)} \quad (11)$$

where $|t| < \min(1, a^{-1})$.

Proof. The last equality in (11) is obvious from the definition of the basic hypergeometric function. Suppose a and x being fixed and put

$$V(t) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} (-xt)^k}{(t; q)_{k+1} (at; q)_{k+1}}.$$

$V(t)$ is a well defined analytic function for $|t| < \min(1, a^{-1})$ which is readily seen to satisfy the q -difference equation

$$(1-t)(1-at)V(t) = 1 - xtV(qt). \quad (12)$$

Writing the power series expansion of $V(t)$ at $t = 0$ in the form

$$V(t) = \sum_{n=0}^{\infty} u_n q^{n(n-1)/2} (-t)^n$$

and inserting the series into (12) one finds that the coefficients u_n obey the recurrence (3) and the initial conditions $u_0 = 1$, $u_1 = -1 - a + x$. Necessarily, $u_n = F_n(a, q; x)$ for all $n \in \mathbb{Z}_+$. \square

In [16, Section 4] and particularly in [17, Eq. (2.6)] there is stated an explicit formula for the polynomials $F_n(a, q; x)$, namely

$$F_n(a, q; x) = (-1)^n q^{-n(n-1)/2} \sum_{j=0}^n \frac{q^{jn} (q^{-n}; q)_j}{(q; q)_j} {}_2\phi_1(q^{j-n}, q^{j+1}; q^{-n}; q, q^{-j}a) x^j.$$

Let us restate this formula as an immediate corollary of Proposition 1.

Corollary 2. *The polynomials $F_n(a, q; x)$, $n \in \mathbb{Z}_+$, can be expressed explicitly as follows*

$$F_n(a, q; x) = (-1)^n q^{-n(n-1)/2} \sum_{j=0}^n \frac{(-1)^j q^{j(j-1)/2}}{(q; q)_j^2} \left(\sum_{k=0}^{n-j} (q^{k+1}; q)_j (q^{n-j-k+1}; q)_j a^k \right) x^j. \quad (13)$$

Proof. The formula can be derived by equating the coefficients of equal powers of t in (11). To this end, one has to apply the q -binomial formula

$$\frac{1}{(z; q)_k} = {}_1\phi_0(q^k; ; q; z) = \sum_{n=0}^{\infty} \frac{(q^k; q)_n}{(q; q)_n} z^n, \quad |z| < 1,$$

cf. [12, Eq. (II.3)]. □

2.3 The indeterminate case and the Nevanlinna parametrization

We are still assuming that a is positive. In [17, Lemma 3.1] it is proved that the Hamburger moment problem for the orthogonal polynomials $F_n(a, q; x)$ (or $P_n(a, q; x)$) is indeterminate if and only if $q < a < q^{-1}$. This is, however, clear from formulas (10) and from the well known criterion (cf. Addenda and Problems 10. to Chapter 2 in [2]) according to which the Hamburger moment problem is indeterminate if and only if

$$\sum_{n=0}^{\infty} (P_n(a, q; 0)^2 + Q_n(a, q; 0)^2) < \infty.$$

This also means that the Jacobi matrix operator J defined in (8), (7), with $\text{Dom } J$ equal to the linear hull of the canonical basis in $\ell^2(\mathbb{Z}_+)$, is not essentially self-adjoint if and only if $a \in (q, q^{-1})$. In this case, the deficiency indices of J are $(1, 1)$, see [2, Chapter 4].

Hence for $q < a < q^{-1}$ there exist infinitely many distinct measures of orthogonality parametrized with the aid of the Nevanlinna functions A , B , C , and D ,

$$\begin{aligned} A(z) &= z \sum_{n=0}^{\infty} Q_n(0) Q_n(z), & B(z) &= -1 + z \sum_{n=0}^{\infty} Q_n(0) P_n(z), \\ C(z) &= 1 + z \sum_{n=0}^{\infty} P_n(0) Q_n(z), & D(z) &= z \sum_{n=0}^{\infty} P_n(0) P_n(z), \end{aligned}$$

where P_n and Q_n are the polynomials of the first and second kind, respectively [2, 22]. All these Nevanlinna functions are entire and

$$A(z)D(z) - B(z)C(z) = 1, \quad \forall z \in \mathbb{C}. \quad (14)$$

According to the Nevanlinna theorem, all measures of orthogonality μ_φ for which the set $\{P_n; n \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}, d\mu_\varphi)$, are in one-to-one correspondence

with functions φ belonging to the one-point compactification $\mathcal{P} \cup \{\infty\}$ of the space of Pick functions \mathcal{P} . Recall that Pick functions are defined and holomorphic on the open complex halfplane $\text{Im } z > 0$, with values in the closed halfplane $\text{Im } z \geq 0$. The correspondence is established by identifying the Stieltjes transform of the measure μ_φ ,

$$\int_{\mathbb{R}} \frac{d\mu_\varphi(x)}{z-x} = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (15)$$

By a theorem due to M. Riesz, $\{P_n; n \in \mathbb{Z}_+\}$ is an orthonormal basis in $L^2(\mathbb{R}, d\mu_\varphi)$ if and only if $\varphi = t$ is a constant function with $t \in \mathbb{R} \cup \{\infty\}$ [2, Theorem 2.3.3]. Then the measure μ_t is said to be N-extremal. Moreover, the N-extremal measures μ_t are in one-to-one correspondence with the self-adjoint extensions T_t of the Jacobi operator J mentioned above. In more detail, if E_t is the spectral measure of T_t and e_0 is the first vector of the canonical basis in $\ell^2(\mathbb{Z}_+)$ then $\mu_t = \langle e_0, E_t(\cdot)e_0 \rangle$ [2, Chapter 4]. The operators T_t in the indeterminate case are known to have a compact resolvent. Hence any N-extremal measure μ_t is purely discrete and supported on $\text{spec } T_t$.

On the other hand, referring to (15), the support of μ_t is also known to be equal to the zero set

$$\mathfrak{Z}_t = \{x \in \mathbb{R}; B(x)t - D(x) = 0\} \quad (16)$$

[2, Section 2.4]. Hence

$$\mu_t = \sum_{x \in \mathfrak{Z}_t} \rho(x) \delta_x \quad (17)$$

where $\rho(x) = \mu_t(\{x\})$ and δ_x is the Dirac measure supported on $\{x\}$. Equation (15), with $\varphi = t$, is nothing but the Mittag-Leffler expansion of the meromorphic function on the right-hand side,

$$\sum_{x \in \mathfrak{Z}_t} \frac{\rho(x)}{z-x} = \frac{A(z)t - C(z)}{B(z)t - D(z)},$$

cf. [2, footnote on p. 55]. From here it can be deduced that

$$\rho(x) = \text{Res}_{z=x} \frac{A(z)t - C(z)}{B(z)t - D(z)} = \frac{A(x)t - C(x)}{B'(x)t - D'(x)} = \frac{1}{B'(x)D(x) - B(x)D'(x)} \quad (18)$$

since, for $x \in \mathfrak{Z}_t$, $t = D(x)/B(x)$.

It should be noted that we are dealing with the Stieltjes case for the matrix operator J is positive on its domain of definition, see (9). This means that, for any choice of parameters from the specified range, there always exists a measure of orthogonality with its support contained in $[0, +\infty)$. In particular, if $a \in (q, q^{-1})$ then at least one of the measures of orthogonality is supported by $[0, +\infty)$. From [10, Lemma 1] it is seen that the limit

$$\lim_{n \rightarrow \infty} \frac{P_n(0)}{Q_n(0)} = \alpha \in (-\infty, 0] \quad (19)$$

exists. And, as explained in [5, Remark 2.2.2], an N-extremal measure of orthogonality μ_t is supported by $[0, \infty)$ if and only if $t \in [\alpha, 0]$, the Stieltjes moment problem is

determinate for $\alpha = 0$ and indeterminate for $\alpha < 0$. Let us note that μ_0 is the unique N-extremal measure for which 0 is a mass point.

In our case, making once more use of the explicit form (10), we have

$$\alpha = \lim_{n \rightarrow \infty} \frac{P_n(a, q; 0)}{Q_n(a, q; 0)} = \begin{cases} -1, & \text{if } a \in (0, 1], \\ -a, & \text{if } a > 1. \end{cases} \quad (20)$$

Hence the Stieltjes problem is indeterminate for any value $a \in (q, q^{-1})$.

The self-adjoint operator T_α corresponding to the N-extremal measure μ_α is nothing but the Friedrichs extension of J [21, Proposition 3.2]. The parameter α can also be computed in the limit

$$\alpha = \lim_{x \rightarrow -\infty} \frac{D(x)}{B(x)},$$

and by inspection of the function $D(x)/B(x)$ one finds that μ_t has exactly one negative mass point if $t \notin [\alpha, 0]$ including $t = \infty$ [5, Lemma 2.2.1]. It is known, too, that Markov's theorem applies in the indeterminate Stieltjes case meaning that

$$\lim_{n \rightarrow \infty} \frac{Q_n(z)}{P_n(z)} = \frac{A(z)\alpha - C(z)}{B(z)\alpha - D(z)}, \quad z \in \mathbb{C} \setminus \text{supp}(\mu_\alpha) \quad (21)$$

[6, Theorem 2.1]. In addition, in the same case, one has the limit

$$\lim_{n \rightarrow \infty} \frac{P_n(z)}{Q_n(0)} = D(z) - B(z)\alpha, \quad z \in \mathbb{C}, \quad (22)$$

as derived in [10] and also in [21].

Further we wish to recall yet another interesting application of the Nevanlinna functions. It is shown in [8] that the reproducing kernel can be expressed in terms of functions $B(z)$ and $D(z)$,

$$K(u, v) := \sum_{n=0}^{\infty} P_n(u)P_n(v) = \frac{B(u)D(v) - D(u)B(v)}{u - v}, \quad (23)$$

see also [7, Section 1].

Finally, let us note that Krein considered a slightly different parametrization of the set of solutions to an indeterminate Stieltjes moment problem, see [19, Chapter V, §5]. The Krein parametrization uses four entire functions a , b , c , and d which can be expressed in terms of the Nevanlinna functions,

$$\begin{aligned} a(z) &= A(-z) - \frac{C(-z)}{\alpha}, & b(z) &= -B(-z) + \frac{D(-z)}{\alpha}, \\ c(z) &= C(-z), & d(z) &= -D(-z), \end{aligned} \quad (24)$$

with α being defined in (19); see also [7, Section 5]. The solutions μ_σ are in one-to-one correspondence with functions σ from the one-point compactification of the Stieltjes class \mathcal{S}^- , see [19] for details. The correspondence is established by the formula

$$\int_0^\infty \frac{d\mu_\sigma(x)}{z + x} = \frac{a(z) + c(z)\sigma(z)}{b(z) + d(z)\sigma(z)}, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $\sigma \in \mathcal{S}^- \cup \{\infty\}$.

2.4 An explicit form of the Nevanlinna functions

In order to describe conveniently the Nevanlinna parametrization in the studied case we introduce a shorthand notation for particular basic hypergeometric series while not indicating the dependance on q explicitly. We put

$$\varphi_a(z) = {}_1\phi_1(0; qa; q, z), \quad \psi_a(z) = {}_1\phi_1(0; qa^{-1}; q, a^{-1}z), \quad (25)$$

and

$$\chi_1(z) = \left. \frac{\partial}{\partial p} {}_1\phi_1(0; p; q, z) \right|_{p=q}.$$

Theorem 3. *Let $1 \neq a \in (q, q^{-1})$. Then the entire functions A , B , C and D from the Nevanlinna parametrization are as follows:*

$$\begin{aligned} A(a, q; z) &= \frac{\varphi_a(qz) - \psi_a(qz)}{1 - a}, \quad B(a, q; z) = \frac{a\psi_a(z) - \varphi_a(z)}{1 - a}, \\ C(a, q; z) &= \frac{\psi_a(qz) - a\varphi_a(qz)}{1 - a}, \quad D(a, q; z) = \frac{a(\varphi_a(z) - \psi_a(z))}{1 - a}. \end{aligned} \quad (26)$$

For $a = 1$ these functions take the form

$$\begin{aligned} A(1, q; z) &= -2q \chi_1(qz) - z \frac{\partial}{\partial z} \varphi_1(qz), \quad B(1, q; z) = 2q \chi_1(z) + z^2 \frac{\partial}{\partial z} (z^{-1} \varphi_1(z)), \\ C(1, q; z) &= 2q \chi_1(qz) + \frac{\partial}{\partial z} (z \varphi_1(qz)), \quad D(1, q; z) = -2q \chi_1(z) - z \frac{\partial}{\partial z} \varphi_1(z). \end{aligned} \quad (27)$$

Proof. We shall confine ourselves to computing the function A only. The formulas for B , C and D can be derived in a fully analogous manner. Starting from the definition of A and recalling formulas (10) and (6), (4) for $Q_n(a, q; 0)$ and $Q_n(a, q; x)$, respectively, one has

$$\begin{aligned} A(a, q; z) &= \frac{qz}{1 - a} \sum_{n=1}^{\infty} (-1)^{n+1} (a^{-n} - 1) q^{n(n-1)/2} F_{n-1}(a, q; qz) \\ &= \frac{zq}{1 - a} \left(a^{-1} \sum_{n=0}^{\infty} q^{n(n-1)/2} F_n(a, q; qz) (-qa^{-1})^n - \sum_{n=0}^{\infty} q^{n(n-1)/2} F_n(a, q; qz) (-q)^n \right). \end{aligned}$$

From comparison of both sums in the last expression with formula (11) for the generating function it becomes clear that the sums can be expressed in terms of basic hypergeometric functions, namely

$$\begin{aligned} A(a, q; z) &= \frac{qz}{1 - a} \left(\frac{a^{-1} {}_2\phi_2(0, q; q^2 a^{-1}, q^2; q, q^2 a^{-1} z)}{(1 - qa^{-1})(1 - q)} - \frac{{}_2\phi_2(0, q; q^2, aq^2; q, zq^2)}{(1 - q)(1 - qa)} \right) \\ &= \frac{1}{1 - a} \left((1 - {}_1\phi_1(0; qa^{-1}; q, qa^{-1}z)) - (1 - {}_1\phi_1(0; qa; q, qz)) \right). \end{aligned}$$

Thus one arrives at the first equation in (26).

Concerning the particular case $a = 1$, formulas (27) can be derived by applying the limit $a \rightarrow 1$ to formulas (26). This is actually possible since Proposition 2.4.1 and Remark 2.4.2 from [5] guarantee that the functions $A(a, q; z)$, $B(a, q; z)$, $C(a, q; z)$, $D(a, q; z)$ depend continuously on $a \in (q, q^{-1})$. In order to be able to apply this theoretical result one has to note that the coefficients in the recurrence (3) depend continuously on a , and to verify that the series $\sum_{n=0}^{\infty} P_n(a, q; 0)^2$ and $\sum_{n=0}^{\infty} Q_n(a, q; 0)^2$ converge uniformly for a in compact subsets of (q, q^{-1}) . But the latter fact is obvious from (10).

For instance, in case of function A one finds that

$$A(1, q; z) = \lim_{a \rightarrow 1} \frac{\varphi_a(qz) - \psi_a(qz)}{1 - a} = -\frac{\partial}{\partial a}(\varphi_a(qz) - \psi_a(qz)) \Big|_{a=1}.$$

A straightforward computation yields the first equation in (27), and similarly for the remaining three equations. \square

Corollary 4. *The following limits are true:*

$$\begin{aligned} \lim_{n \rightarrow \infty} (-1)^n \left(\frac{a}{q}\right)^{n/2} P_n(a, q; x) &= \frac{{}_1\phi_1(0; qa; q, x)}{1 - a}, & \text{if } q < a < 1, \\ \lim_{n \rightarrow \infty} (-1)^n (qa)^{-n/2} P_n(a, q; x) &= \frac{a {}_1\phi_1(0; qa^{-1}; q, a^{-1}x)}{a - 1}, & \text{if } 1 < a < q^{-1}, \\ \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} q^{-n/2} P_n(1, q; x) &= {}_1\phi_1(0; q; q, x), \end{aligned} \quad (28)$$

and

$$\lim_{n \rightarrow \infty} \frac{Q_n(z)}{P_n(z)} = \begin{cases} -\frac{{}_1\phi_1(0; qa; q, qz)}{{}_1\phi_1(0; qa; q, z)} & \text{for } q < a < 1, \\ -\frac{{}_1\phi_1(0; qa^{-1}; q, qa^{-1}z)}{a {}_1\phi_1(0; qa^{-1}; q, a^{-1}z)} & \text{for } 1 < a < q^{-1}, \\ -\frac{{}_1\phi_1(0; q; q, qz)}{{}_1\phi_1(0; q; q, z)} & \text{for } a = 1. \end{cases} \quad (29)$$

Proof. From (26), (27) and (20) one immediately infers that

$$\begin{aligned} C(a, q; z) - A(a, q; z)\alpha &= \varphi_a(qz) \text{ or } \psi_a(qz) \text{ or } \varphi_1(qz), \\ D(a, q; z) - B(a, q; z)\alpha &= -\varphi_a(z) \text{ or } -a\psi_a(z) \text{ or } -\varphi_1(z), \end{aligned}$$

depending on whether $q < a < 1$ or $1 < a < q^{-1}$ or $a = 1$. Equations (28) follow from (22) and (10) while equations (29) are a direct consequence of (21). \square

Remark 5. The limits (28) can be proved, in an alternative way, by applying Darboux's method to the generating function whose explicit form is given in (11). According to this method, the leading asymptotic term of $q^{n(n-1)/2} F_n(a, q; x)$ is determined by the singularity of the function on the left-hand side in (11) which is located most closely

to the origin, cf. [20, Section 8.9]. Proceeding this way one can show that the first limit in (28) is valid even for all $0 < a < 1$ while the second one is valid for all $a > 1$. Let us also note that the limits established in (28) can be interpreted as a q -analogue to Hurwitz's limit formula for the Lommel polynomials. The case $a < 1$ has been derived, probably for the first time, in [16, Eq. (4.24)], see also [15, Eq. (3.4)] and [17, Eq. (2.7)], while the case $a > 1$ has been treated in [15, Eq. (3.6)].

The following formula for the reproducing kernel can be established.

Corollary 6. *Suppose $q < a < q^{-1}$. Then*

$$K(u, v) = \frac{a(\varphi_a(u)\psi_a(v) - \psi_a(u)\varphi_a(v))}{(1-a)(u-v)}$$

if $a \neq 1$, and

$$K(u, v) = \frac{\varphi_1(u)(2q\chi_1(v) + v\varphi_1'(v)) - (2q\chi_1(u) + u\varphi_1'(u))\varphi_1(v)}{u-v}$$

if $a = 1$.

Proof. This is a direct consequence of (23) and (26), (27). \square

Remark 7. In [24], the self-adjoint extensions of the Jacobi matrix J , defined in (7), (8), are described in detail while addressing only the case $q < a < 1$. The self-adjoint extensions, called $T(\kappa)$, are parameterized by $\kappa \in \mathbb{R} \cup \{\infty\}$, with $\kappa = \infty$ corresponding to the Friedrichs extension. The domain $\text{Dom } T(\kappa) \subset \text{Dom } J^*$ is specified by the asymptotic boundary condition: a sequence f from $\text{Dom } J^*$ belongs to $\text{Dom } T(\kappa)$ if and only if $C_2(f) = \kappa C_1(f)$ where

$$C_1(f) = \lim_{n \rightarrow \infty} (-1)^n \left(\frac{a}{q}\right)^{n/2} f_n, \quad C_2(f) = \lim_{n \rightarrow \infty} \left((-1)^n f_n - C_1(f) \left(\frac{q}{a}\right)^{n/2}\right) (qa)^{-n/2}$$

(the limits can be shown to exist). The eigenvalues of $T(\kappa)$ are exactly the roots of the equation

$$\kappa {}_1\phi_1(0; qa; q, x) + a {}_1\phi_1(0; qa^{-1}; q, a^{-1}x) = 0.$$

On the other hand, consider the self-adjoint extension T_t corresponding the measure of orthogonality μ_t , with $t \in \mathbb{R} \cup \{\infty\}$ being a Nevanlinna parameter. The eigenvalues of T_t are the mass points from the support of μ_t , i.e. the zeros of the function

$$(1-a)(B(a, q; x)t - D(a, q; x)) = (t+1)a {}_1\phi_1(0; qa^{-1}; q, a^{-1}x) - (t+a) {}_1\phi_1(0; qa; q, x),$$

as one infers from (25) and (26). Since a self-adjoint extension is unambiguously determined by its spectrum (see, for instance, proof of Theorem 4.2.4 in [2]) one gets the correspondence $\kappa = \Upsilon(t)$ where

$$\Upsilon(t) = -\frac{a+t}{1+t}. \quad (30)$$

Remark 8. Note that if the Krein parametrization (24) is used in Theorem 3 rather than the Nevanlinna parametrization, some parameter functions acquire particularly simple form. For instance, $a(z) = \varphi_a(-qz)$ and $b(z) = \varphi_a(-z)$ provided $q < a < 1$. This simplification does not apply for functions c and d , however.

Being motivated by this observation we propose yet another parametrization which is well suited to our problem. For $1 \neq a \in (q, q^{-1})$, let us put

$$\begin{aligned}\mathcal{A}(a, q; z) &= A(a, q; z) + C(a, q; z), & \mathcal{B}(a, q; z) &= -B(a, q; z) - D(a, q; z), \\ \mathcal{C}(a, q; z) &= aA(a, q; z) + C(a, q; z), & \mathcal{D}(a, q; z) &= -B(a, q; z) - a^{-1}D(a, q; z),\end{aligned}$$

Then $\mathcal{A}(z) = \mathcal{B}(qz) = \varphi_a(qz)$ and $\mathcal{C}(z) = \mathcal{D}(qz) = \psi_a(qz)$. Moreover, we have

$$\frac{A(a, q; z)\varphi(z) - C(a, q; z)}{B(a, q; z)\varphi(z) - D(a, q; z)} = -\frac{\mathcal{A}(a, q; z)\Upsilon(\varphi(z)) + \mathcal{C}(a, q; z)}{\mathcal{B}(a, q; z)\Upsilon(\varphi(z)) + a\mathcal{D}(a, q; z)}$$

where Υ is defined in (30).

Clearly, $\Upsilon \circ \Upsilon = \text{id}$. Assuming $1 < a < q^{-1}$, the composition with $\Upsilon \in \mathcal{P}$ induces a one-to-one mapping of \mathcal{P} onto itself since the set of Pick functions is closed under composition. Thus we obtain a correspondence between solutions μ_ω of the Hamburger moment problem for the polynomial sequence $\{P_n(a, q; z)\}$ and functions $\omega \in \mathcal{P} \cup \{\infty\}$ which is established via the equation

$$\int_{\mathbb{R}} \frac{d\mu_\omega(x)}{z - x} = -\frac{\varphi_a(qz)\omega(z) + \psi_a(qz)}{\varphi_a(z)\omega(z) + a\psi_a(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Similarly, for $q < a < 1$, we have $-\Upsilon \in \mathcal{P}$ and an analogous correspondence can be established in the form

$$\int_{\mathbb{R}} \frac{d\mu_\omega(x)}{z - x} = -\frac{\psi_a(qz)\omega(z) + \varphi_a(qz)}{a\psi_a(z)\omega(z) + \varphi_a(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Remark 9. There is a close connection between this work and an earlier research conducted by Chen and Ismail [9] who studied the indeterminate moment problem associated with the orthogonal polynomial sequence $\{P_n\}_{n=0}^\infty$ determined by the recurrence

$$zP_n(z) = q^{-n-1}P_{n+1}(z) + q^{-n}P_{n-1}(z) \quad \text{for } n \geq 1,$$

with $P_0(z) = 1$, $P_1(z) = qz$. These polynomials are a particular case of the polynomials $P_n(a, q; x)$ defined in (5). Indeed, for all $n \in \mathbb{Z}_+$,

$$P_n(z) = P_n(-1, q; i\sqrt{q}z).$$

For this particular choice of parameters one can deduce from (11) that a generating function for the polynomials $\{P_n(z)\}$ can be expressed in the form

$$\sum_{n=0}^{\infty} P_n(z)t^n = \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2} (zt)^k}{(-qt^2; q^2)_{k+1}}$$

coinciding with a formula obtained by Chen and Ismail, see [9, Theorem 3.1]. Note, however, that a , being specialized to -1 , is out of the range considered in the present paper.

With the knowledge of the generating function one can proceed similarly as in Theorem 3 reproducing this way a result concerned with the Nevanlinna parametrization which has been obtained by Chen and Ismail in [9, Theorem 3.2]. Let us denote by \tilde{A} , \tilde{B} , \tilde{C} , and \tilde{D} the Nevanlinna functions associated with the polynomials $P_n(z)$. Using the notation introduced in (26) we have

$$\begin{aligned}\tilde{A}(z) &= iq^{1/2}A(-1, q; iq^{1/2}z), & \tilde{B}(z) &= B(-1, q; iq^{1/2}z), \\ \tilde{C}(z) &= C(-1, q; iq^{1/2}z), & \tilde{D}(z) &= -iq^{-1/2}D(-1, q; iq^{1/2}z).\end{aligned}$$

Let us remark that $\varphi_{-1}(z) = \psi_{-1}(-z)$. This makes it possible to express \tilde{A} , \tilde{B} , \tilde{C} , and \tilde{D} in a comparatively simple form. For instance, a straightforward computation yields

$$\tilde{A}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2(n+1)^2}}{(q^2; q^2)_{2n+1}} z^{2n+1}.$$

2.5 Measures of orthogonality

With the explicit knowledge of the Nevanlinna parametrization established in Theorem 3 it is straightforward to describe all N -extremal solutions.

Proposition 10. *Let $1 \neq a \in (q, q^{-1})$. Then all N -extremal measures $\mu_t = \mu_t(a, q)$, $t \in \mathbb{R} \cup \{\infty\}$, are of the form*

$$\mu_t = \sum_{x \in \mathfrak{Z}_t} \rho(x) \delta_x \quad \text{where} \quad \frac{1}{\rho(x)} = \frac{a}{1-a} (\psi_a(x) \varphi'_a(x) - \varphi_a(x) \psi'_a(x)), \quad (31)$$

$$\mathfrak{Z}_t = \mathfrak{Z}_t(a, q) = \{x \in \mathbb{R}; a(t+1)\psi_a(x) - (t+a)\varphi_a(x) = 0\},$$

and δ_x stands for the Dirac measure supported on $\{x\}$.

For $a = 1$, all N -extremal measures $\mu_t = \mu_t(1, q)$ are of the form $\mu_t = \sum_{x \in \mathfrak{Y}_t} \rho(x) \delta_x$ where

$$\frac{1}{\rho(x)} = 2q(\varphi'_1(x)\chi_1(x) - \varphi_1(x)\chi'_1(x)) + x(\varphi'_1(x))^2 - \varphi_1(x)\varphi'_1(x) - x\varphi_1(x)\varphi''_1(x).$$

and

$$\mathfrak{Y}_t = \mathfrak{Y}_t(q) = \{x \in \mathbb{R}; 2q(t+1)\chi_1(x) + (t+1)x\varphi'_1(x) - t\varphi_1(x) = 0\}.$$

Proof. Referring to general formulas (17) and (16), (18), it suffices to apply Theorem 3. \square

The formula for the orthogonality measure μ_t , as given in (31), is far of being explicit. To elucidate its structure one can attempt at least to provide the asymptotics for the mass points $x \in \mathfrak{Z}_t$ and the weights $\rho(x)$ for x large. This is actually possible owing to the fact that a good deal of attention has been paid by various authors to the study of the asymptotic behavior of the roots of the q -Bessel functions. We have postponed to Appendix a summary of basic facts and references on this subject that we need for our purposes. Moreover, these facts are completed therein with some additional details. In the asymptotic analysis to follow we restrict our attention to the range of parameters $0 < q < a < 1$ only.

Suppose $t \in \mathbb{R} \cup \{\infty\}$. As pointed out in Remark 7, the mass points of the measure of orthogonality μ_t are, at the same time, eigenvalues of the corresponding self-adjoint extension T_t of the matrix (Jacobi) operator J introduced in (8). The eigenvalues are solutions to the characteristic equation $\Phi_t(x) = 0$ where

$$\begin{aligned}\Phi_t(x) &= a(t+1)\psi_a(x) - (t+a)\varphi_a(x) \\ &= a(t+1) {}_1\phi_1\left(0; \frac{q}{a}; q, \frac{x}{a}\right) - (t+a) {}_1\phi_1(0; aq; q, x),\end{aligned}\quad (32)$$

they are all real and simple, and their only accumulation point is $+\infty$. If x is an eigenvalue of T_t then the root x of $\Phi_t(x)$ is also simple. Actually, this is a standard fact following from the discrete Green-like formula implying that the square norm of the corresponding eigenvector is proportional to $\Phi'_t(x)$ (see [2, § I.2.1] or, for instance, [23, Lemma 4]).

Let us order the mass points of the measure μ_t , i.e. the roots of $\Phi_t(x)$, increasingly, $\xi_0^{(t)} < \xi_1^{(t)} < \xi_2^{(t)} < \dots$. More is known about the properties of these roots, see § 2.4 in [24] for additional details. Remember T_{-1} coincides with the Friedrichs extension of J . For any fixed $m \in \mathbb{Z}_+$ and using the substitution $t = -(\kappa + a)/(\kappa + 1)$, one knows that thus obtained composed function $\xi_m^{(t)}$ is strictly monotonic increasing in the parameter $\kappa \in \mathbb{R}$. Furthermore, for any $t \in \mathbb{R} \cup \{\infty\}$, $t \neq -1$, we have the inequalities

$$\xi_0^{(t)} < \xi_0^{(-1)} \text{ and } \xi_{m-1}^{(-1)} < \xi_m^{(t)} < \xi_m^{(-1)} \text{ for } m \geq 1.$$

Moreover, Proposition A.3 from Appendix tells us that

$$\xi_m^{(-1)} = q^{-m} + O(a^m q^{m(m+1)}) \text{ and } \xi_m^{(-a)} = aq^{-m} + O(a^{-m} q^{m(m+1)}) \text{ as } m \rightarrow \infty. \quad (33)$$

For a general t we have the following result.

Proposition 11. *Suppose $0 < q < a < 1$ and let $t \in \mathbb{R} \cup \{\infty\}$, $t \neq -1$. Then*

$$\xi_m^{(t)} = aq^{-m} \left(1 - \frac{(1-a)(t+a)}{t+1} \frac{(q/a; q)_\infty^2}{(q; q)_\infty^2} a^{m-1} + O(a^{2m}) \right) \quad (34)$$

and

$$\frac{1}{\rho(\xi_m^{(t)})} = (q; q)_\infty^2 a^m q^{-m^2} (1 + O(ma^m)) \quad (35)$$

as $m \rightarrow \infty$.

Proof. We seek a root ξ of the characteristic equation in vicinity of aq^{-m} while writing $\xi = aq^{-m} + \epsilon$, with $|q^m \epsilon| \ll 1$, and assuming m to be sufficiently large. Referring to Theorem A.1 one readily concludes that

$$\frac{A(x)}{A(x/a)} = D(x) \exp\left(-\frac{\ln(a)}{\ln(q)} \ln(x)\right) \text{ where } D(x) = O(1) \text{ as } x \rightarrow +\infty,$$

and therefore, in the considered asymptotic domain, the characteristic equation can be given the form

$$\sin\left(\frac{\pi \ln(x/a)}{\ln(q)}\right) = O(x^{-\ln(a)/\ln(q)}).$$

Hence, for all sufficiently large m , a solution $\xi = \xi(m)$ is sure to exist such that $\xi(m) = aq^{-m} (1 + O(a^m))$ as $m \rightarrow \infty$. To get the second asymptotic term explicitly note that, in view of (A.3),

$$\begin{aligned} \frac{A(aq^{-m})}{A(q^{-m})} &= a^{m+1/2} \exp\left(-\frac{\ln^2(a)}{2\ln(q)}\right) \frac{|\left(\tilde{q}e^{-2i\beta(a)}; \tilde{q}\right)_\infty|^2}{(\tilde{q}; \tilde{q})_\infty^2} \\ &= -\frac{(a; q)_\infty (q/a; q)_\infty}{(q; q)_\infty^2} \frac{\pi a^m}{\ln(q) \sin(\beta(a))}. \end{aligned}$$

Now it is somewhat tedious but straightforward to apply Theorem A.1 to (32) in order to derive that

$$\epsilon(m) = -aq^{-m} \frac{(1-a)(t+a)}{t+1} \frac{(q/a; q)_\infty^2}{(q; q)_\infty^2} a^{m-1} (1 + O(a^m)). \quad (36)$$

Hence the sequence of roots $\{\xi(m)\}$ we have found meets the asymptotic behavior as claimed in the proposition. To conclude the proof it suffices to observe that (36) along with (33) guarantee $\xi(m)$ to be sufficiently close to $\xi_m^{(-a)}$ and, consequently, $\xi_{m-1}^{(-1)} < \xi(m) < \xi_m^{(-1)}$, implying that $\xi(m) = \xi_m^{(t)}$ is actually the m th root of $\Phi_t(x)$.

As far as the weights are concerned, sticking to notation (A.2) and making use of (34) it is straightforward to derive that

$$A(\xi_m^{(t)}) = A(a) a^m q^{-(m+1)m/2} (1 + O(ma^m)).$$

Clearly,

$$\sin(\beta(\xi_m^{(t)})) = (-1)^m \sin(\beta(a)) (1 + O(a^m)).$$

Taking into account (A.3) and Theorem A.1 one finds that

$$\varphi_a(\xi_m^{(t)}) = (-1)^m (q/a; q)_\infty (1-a) a^m q^{-(m+1)m/2} (1 + O(ma^m)).$$

To analyze

$$\psi'_a(\xi_m^{(t)}) = \frac{1}{a} \frac{\partial {}_1\phi_1(0; q/a; q, z)}{\partial z} \Big|_{z=\xi_m^{(t)}/a}$$

for m large one can make use of Theorem A.2 along with (A.7) to obtain

$$\psi'_a(\xi_m^{(t)}) = (-1)^{m+1} \frac{(q; q)_\infty^2}{a (q/a; q)_\infty} q^{-m(m-1)/2} (1 + O(ma^m)).$$

Similarly one finds that $\varphi'_a(\xi_m^{(t)}) = O(ma^m q^{-m(m-1)/2})$. Consequently, in view of (31),

$$\begin{aligned} \frac{1}{\rho(\xi_m^{(t)})} &= \frac{a \varphi_a(\xi_m^{(t)})}{1-a} \left(\frac{t+a}{a(t+1)} \varphi'_a(\xi_m^{(t)}) - \psi'_a(\xi_m^{(t)}) \right) \\ &= -\frac{a}{1-a} \varphi_a(\xi_m^{(t)}) \psi'_a(\xi_m^{(t)}) (1 + O(ma^m)). \end{aligned}$$

Relation (35) readily follows. \square

In what follows we again admit any value of a lying between q and q^{-1} .

Lemma 12. *With the notation introduced in (25) it holds true that*

$$\varphi_a(z)\psi_a(qz) - a\psi_a(z)\varphi_a(qz) = 1 - a \quad \text{if } a \neq 1, \quad (37)$$

and

$$2q(\varphi_1(z)\chi_1(qz) - \chi_1(z)\varphi_1(qz)) + z(q\varphi_1(z)\varphi'_1(qz) - \varphi'_1(z)\varphi_1(qz)) + \varphi_1(z)\varphi_1(qz) = 1,$$

for all $z \in \mathbb{C}$.

Proof. These identities follow from (14) and, again, from Theorem 3. \square

Let us examine a bit more closely two particular N-extremal measures μ_t described in Proposition 10, with $t = -1$ and $t = -a$. They correspond to the distinguished cases $t = \alpha$ if $a \in (q, 1)$ or $a \in (1, q^{-1})$, respectively (cf. (20)). As already mentioned, if $t = \alpha$ then the corresponding self-adjoint extension of the underlying Jacobi matrix is the Friedrichs extension, and the measure μ_t is necessarily a Stieltjes measure. In the case $t = -1$ the orthogonality relation for the orthonormal polynomials $P_n(a, q; x)$ reads

$$-\sum_{k=0}^{\infty} \frac{\varphi_a(q\xi_k)}{\varphi'_a(\xi_k)} P_n(a, q; \xi_k) P_m(a, q; \xi_k) = \delta_{mn} \quad (38)$$

where $\{\xi_k; k \in \mathbb{Z}_+\}$ are the zeros of the function φ_a . Actually, from (31) and (37) one infers that $\rho(\xi_k) = -\varphi_a(q\xi_k)/\varphi'_a(\xi_k)$ if $\varphi_a(\xi_k) = 0$. Similarly, the same orthogonality relation for $t = -a$ reads

$$-\frac{1}{a} \sum_{k=0}^{\infty} \frac{\psi_a(q\eta_k)}{\psi'_a(\eta_k)} P_n(a, q; \eta_k) P_m(a, q; \eta_k) = \delta_{mn} \quad (39)$$

where $\{\eta_k; k \in \mathbb{Z}_+\}$ are the zeros of the function ψ_a . One can refer once more to [3, Thm. 2.2, Rem. 2.3] where it is shown (if rewritten in our notation) that

$$\xi_k = q^{-k} (1 + O(q^k)) \quad \text{and} \quad \eta_k = aq^{-k} (1 + O(q^k)) \quad \text{as } k \rightarrow \infty.$$

Remark 13. The orthogonality relation (38) has been derived already in [17, Theorem 3.6.]. This is the unique orthogonality relation for the polynomials $P_n(a, q; x)$ if $a \in (0, q]$ (the determinate case), and an example of an N-extremal orthogonality relation if $a \in (q, 1)$. Similarly, (39) is the unique orthogonality relation if $a \geq q^{-1}$. Of course, (38) and (39) coincide for $a = 1$.

Remark 14. In [4, Section 1], an explicit expression has been found for the measures of orthogonality μ_φ corresponding to constant Pick functions $\varphi(z) = \beta + i\gamma$, with $\beta \in \mathbb{R}$ and $\gamma > 0$. Let us call these measures $\mu_{\beta, \gamma} = \mu_{\beta, \gamma}(a, q)$. It turns out that $\mu_{\beta, \gamma}$ is an absolutely continuous measure supported on \mathbb{R} with the density

$$\frac{d\mu_{\beta, \gamma}}{dx} = \frac{\gamma}{\pi} ((\beta B(a, q; x) - D(a, q; x))^2 + \gamma^2 B(a, q; x)^2)^{-1}.$$

In our case, referring to (26), (27), we get the probability density

$$\frac{d\mu_{\beta, \gamma}}{dx} = \frac{\gamma(1-a)^2}{\pi \left(((\beta+1)a\psi_a(x) - (\beta+a)\varphi_a(x))^2 + \gamma^2(a\psi_a(x) - \varphi_a(x))^2 \right)},$$

provided $1 \neq a \in (q, q^{-1})$, and

$$\begin{aligned} \frac{d\mu_{\beta, \gamma}}{dx} &= \frac{\gamma}{\pi} \\ &\times \left((2q(\beta+1)\chi_1(x) - \beta\varphi_1(x) + (\beta+1)x\varphi_1'(x))^2 + \gamma^2(2q\chi_1(x) - \varphi_1(x) + x\varphi_1'(x))^2 \right)^{-1}, \end{aligned}$$

provided $a = 1$. Letting $\beta = -1$ or $\beta = -a$ and $\gamma > 0$ arbitrary, one obtains comparatively simple and nice orthogonality relations for the polynomials $P_n(a, q; x)$, namely

$$\int_{\mathbb{R}} \frac{P_m(a, q; x)P_n(a, q; x)}{\gamma(a\psi_a(x) - \varphi_a(x))^2 + \gamma^{-1}(a-1)^2\varphi_a(x)^2} dx = \frac{\pi}{(a-1)^2} \delta_{mn}$$

and

$$\int_{\mathbb{R}} \frac{P_m(a, q; x)P_n(a, q; x)}{\gamma(a\psi_a(x) - \varphi_a(x))^2 + \gamma^{-1}(a-1)^2a^2\psi_a(x)^2} dx = \frac{\pi}{(a-1)^2} \delta_{mn},$$

valid for all $m, n \in \mathbb{Z}_+$ and $a \in (q, q^{-1})$, $a \neq 1$. If $a = 1$, a similar orthogonality relation takes the form

$$\int_{\mathbb{R}} \frac{P_m(1, q; x)P_n(1, q; x)}{\gamma(2q\chi_1(x) + x\varphi_1'(x) - \varphi_1(x))^2 + \gamma^{-1}\varphi_1(x)^2} dx = \pi\delta_{mn}.$$

3 The moment sequence

3.1 Passing to the determinate case

Let μ be any measure of orthogonality for the orthonormal polynomials $P_n(a, q; x)$ introduced in (5). Denote by

$$m_n(a, q) = \int_{\mathbb{R}} x^n d\mu(x), \quad n \in \mathbb{Z}_+,$$

the corresponding moment sequence. It is clear from Favard's theorem, however, that the moments do not depend on the particular choice of the measure of orthogonality. It is even known that

$$m_n(a, q) = \langle e_0, J(a, q)^n e_0 \rangle, \quad n \in \mathbb{Z}_+, \quad (40)$$

where $J(a, q)$ is the Jacobi matrix defined in (7), (8), and e_0 is the first vector of the canonical basis in $\ell^2(\mathbb{Z}_+)$. Whence $m_n(a, q)$ is a polynomial in a and q^{-1} . Consequently, in order to compute the moments one can admit a wider range of parameters than that we were using up to now, namely $0 < q < 1$ and $q < a < q^{-1}$. This observation can be of particular importance for the parameter q since the properties of the matrix operator $J(a, q)$ would change dramatically if q was allowed to take values $q > 1$. We wish to stick, however, to the widely used convention according to which the modulus of q is smaller than 1. This is why we replace the symbol q by p in this section whenever this restriction is relaxed. Concerning the parameter a , it is always supposed to be positive.

Put, for $p > 0$ and $a > 0$,

$$\omega_n(a, p) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_p p^{-k(n-k)} a^k, \quad n \in \mathbb{Z}_+. \quad (41)$$

The meaning of the q -binomial coefficient in (41) is the standard one, cf. [12, Eq. (I.39)]. Let us remark that $\omega_n(a, p)$ can be expressed in terms of the continuous q -Hermite polynomials $H_n(x; q)$, namely

$$\omega_n(a, p) = {}_2\phi_0(p^n, 0; ; p^{-1}, p^{-n}a) = a^{n/2} H_n\left(\frac{1}{2}(a^{1/2} + a^{-1/2}); p^{-1}\right), \quad (42)$$

see [14].

As before, the monic polynomials $F_n(a, p; x)$ are generated by the recurrence (3), with $F_{-1}(a, p; x) = 0$ and $F_0(a, p; x) = 1$ (writing p instead of q). The following proposition is due to Van Assche and is contained in [25, Theorem 2].

Proposition 15. *For $p > 1$ and $x \neq 0$ one has*

$$\lim_{n \rightarrow \infty} x^{-n} F_n(a, p; x) = \sum_{k=0}^{\infty} \frac{\omega_k(a, p)}{(p; p)_k} \left(\frac{p}{x}\right)^k.$$

Note that if $p > 1$ then the Jacobi matrix $J(a, p)$ represents a compact (even trace class) operator on $\ell^2(\mathbb{Z}_+)$. In particular, this implies that the Hamburger moment problem is determinate. Several additional useful facts are known in this case which we summarize in the following remark.

Remark 16. In [23, Section 3] it is noted that if $\{\beta_n\}_{n=0}^{\infty}$ is a real sequence belonging to $\ell^1(\mathbb{Z}_+)$, $\{\alpha_n\}_{n=0}^{\infty}$ is a positive sequence belonging to $\ell^2(\mathbb{Z}_+)$ and $\{F_n(x)\}_{n=0}^{\infty}$ is a sequence of monic polynomials defined by the recurrence

$$F_{n+1}(x) = (x - \beta_n)F_n(x) - \alpha_{n-1}^2 F_{n-1}(x), \quad n \geq 0,$$

with $F_0(x) = 1$ and (conventionally) $F_{-1}(x) = 0$, then

$$\lim_{n \rightarrow \infty} x^{-n} F_n(x) = \mathcal{G}(x^{-1}) \quad \text{for } x \neq 0 \quad (43)$$

where $\mathcal{G}(z)$ is an entire function. Moreover, let μ be the (necessarily unique) measure of orthogonality for the sequence of polynomials $\{F_n(x)\}$. Then the Stieltjes transform of μ reads

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1 - zx} = \frac{\tilde{\mathcal{G}}(z)}{\mathcal{G}(z)} \quad (44)$$

where $\tilde{\mathcal{G}}(z)$ is an entire function associated in an analogous manner with the shifted sequences $\{\tilde{\alpha}_n = \alpha_{n+1}\}_{n=0}^{\infty}$, $\{\tilde{\beta}_n = \beta_{n+1}\}_{n=0}^{\infty}$.

Theorem 17. *Let $p > 1$ and $x \neq 0$. Then*

$$\lim_{n \rightarrow \infty} x^{-n} F_n(a, p; x) = \mathcal{G}(x^{-1})$$

where

$$\mathcal{G}(z) = (z; p^{-1})_{\infty} {}_1\phi_1(0; z; p^{-1}, az) = \sum_{k=0}^{\infty} \frac{\omega_k(a, p)}{(p; p)_k} (pz)^k \quad (45)$$

is an entire function obeying the second-order q -difference equation

$$\mathcal{G}(z) - (1 - (a + 1)z)\mathcal{G}(p^{-1}z) + ap^{-1}z^2\mathcal{G}(p^{-2}z) = 0. \quad (46)$$

The Stieltjes transform of the (unique) measure of orthogonality μ for the sequence of orthogonal polynomials $\{F_n(a, p; x)\}$ is given by the formula

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1 - zx} = \frac{\mathcal{G}(p^{-1}z)}{\mathcal{G}(z)}. \quad (47)$$

Proof. In view of Proposition 15, in order to show (45) it suffices to verify only the second equality. But this equality follows from the definition of the basic hypergeometric series and from the well known identity [12, Eq. (II.2)]

$$(z; p^{-1})_{\infty} = \sum_{n=0}^{\infty} \frac{(pz)^n}{(p; p)_n}.$$

Using the power series expansion of $\mathcal{G}(z)$ established in (45) one finds that (46) is equivalent to

$$\omega_k - (a + 1)\omega_{k-1} + a(1 - p^{-k+1})\omega_{k-2} = 0 \quad \text{for } k \geq 2$$

and $\omega_1 - (a + 1)\omega_0 = 0$. This is true, indeed, if we take into account (42) and the recurrence relation for the continuous q -Hermite polynomials [14, Eq. (14.26.3)]

$$2xH_k(x; q) = H_{k+1}(x; q) + (1 - q^k)H_{k-1}(x; q).$$

Recalling once more (3), the polynomials $F_n(a, p; x)$ solve the recurrence relation

$$u_{n+1} = (x - (a+1)p^{-n})u_n - ap^{-2n+1}u_{n-1} \quad (48)$$

while the polynomials $\tilde{F}_n(a, p; x) := p^{-n}F_n(a, p; px)$ obviously obey the recurrence

$$\tilde{u}_{n+1} = (x - (a+1)p^{-n-1})\tilde{u}_n - ap^{-2n-1}\tilde{u}_{n-1}. \quad (49)$$

Comparing these two equations one observes that (49) is obtained from (48) just by shifting the index. In other words, the sequences of monic polynomials $\{\tilde{F}_n(a, p; x)\}$ and $\{F_n(a, p; x)\}$ are generated by the same recurrence relation, but the index has to be shifted in the latter case. Hence, referring to Remark 16 and equation (43), one can compute

$$\tilde{\mathcal{G}}(x^{-1}) = \lim_{n \rightarrow \infty} x^{-n} \tilde{F}_n(a, p; x) = \lim_{n \rightarrow \infty} (px)^{-n} F_n(a, p; px) = \mathcal{G}(p^{-1}x^{-1}).$$

Thus $\tilde{\mathcal{G}}(z) = \mathcal{G}(p^{-1}z)$ and (47) is a particular case of (44). \square

3.2 Recurrence relations and asymptotic behavior

From (40) it is seen that $m_n(a, p) \leq \|J(a, p)\|^n$. Moreover, from (47) one deduces that

$$\sum_{n=0}^{\infty} m_n(a, p) z^n = \frac{\mathcal{G}(p^{-1}z)}{\mathcal{G}(z)}, \quad (50)$$

and the series is clearly convergent if $p > 1$ and $|z| < \|J(a, p)\|^{-1}$.

Remark 18. Any explicit formula for monic polynomials $F_n(x)$, $n \in \mathbb{Z}_+$, which are members of a sequence of orthogonal polynomials with a measure of orthogonality μ , automatically implies a linear recursion for the corresponding moments. In fact, $F_0(x) = 1$ and so, by orthogonality, $\int_{\mathbb{R}} F_n(x) d\mu(x) = 0$ for $n \geq 1$. Particularly, in our case, formula (13) implies the relation

$$\sum_{j=0}^n \frac{(-1)^j q^{(j-1)j/2}}{(q; q)_j^2} \left(\sum_{k=0}^{n-j} (q^{k+1}; q)_j (q^{n-j-k+1}; q)_j a^k \right) m_j(a, q) = 0 \quad \text{for } n \geq 1.$$

Further we derive two more recursions for the moments, a linear and a quadratic one.

Proposition 19. *The moment sequence $\{m_n(a, q)\}$ solves the equations $m_0(a, q) = 1$ and*

$$m_n(a, q) = \frac{\omega_n(a, q)}{(q; q)_{n-1}} - \sum_{k=1}^{n-1} \frac{q^k \omega_k(a, q)}{(q; q)_k} m_{n-k}(a, q), \quad n \in \mathbb{N}. \quad (51)$$

Proof. Equations (50) and (45) imply that

$$\sum_{m=0}^{\infty} \frac{p^m \omega_m(a, p)}{(p; p)_m} z^m \sum_{n=0}^{\infty} m_n(a, p) z^n = \sum_{m=0}^{\infty} \frac{\omega_m(a, p)}{(p; p)_m} z^m$$

holds for $p > 1$ and z from a neighborhood of 0. Equating the coefficients of equal powers of z one finds that (51) holds true for $q = p > 1$. But for the both sides are rational functions in q the equation remains valid also for $0 < q < 1$. \square

Proposition 20. *The moment sequence $\{m_n(a, q)\}$ solves the equations $m_0(a; q) = 1$ and*

$$m_{n+1}(a, q) = (a + 1) m_n(a, q) + a \sum_{k=0}^{n-1} q^{-k-1} m_k(a, q) m_{n-k-1}(a, q), \quad n \in \mathbb{Z}_+. \quad (52)$$

Proof. Equation (46) can be rewritten as

$$\frac{\mathcal{G}(p^{-1}z)}{\mathcal{G}(z)} \left(1 - (a + 1)z - ap^{-1}z^2 \frac{\mathcal{G}(p^{-2}z)}{\mathcal{G}(p^{-1}z)} \right) = 1$$

and holds true for $p > 1$ and z from a neighborhood of the origin. Substituting the power series expansion (50) one has

$$\left(1 - (a + 1)z - ap^{-1}z^2 \sum_{n=0}^{\infty} m_n(a, p) p^{-n} z^n \right) \sum_{n=0}^{\infty} m_n(a, p) z^n = 1.$$

Equating the coefficients of equal powers of z one concludes that (52) holds for $q = p > 1$. For the both sides are polynomials in q^{-1} the equation is valid for $0 < q < 1$ as well. \square

Our final task is to provide estimates bringing some insight into the asymptotic behavior of the moments for large powers. We still assume that $0 < q < 1$ and $a > 0$. On the other hand, a is not required to be restricted to the interval $q < a < q^{-1}$. Let us note that it has been shown in [5, Lemma 4.9.1] that

$$a^{n/2} q^{-n(n-1)/4} \leq \omega_n(a, q) \leq (1 + a)^n q^{-n^2/4}, \quad n \in \mathbb{Z}_+. \quad (53)$$

Proposition 21. *Let $a > 0$. The moments $m_n(a, q)$ obey the inequalities*

$$m_n(a, q) \leq \frac{(1 + a)^n}{(q; q)_{n-1}} q^{-n^2/4}, \quad n \in \mathbb{Z}_+, \quad (54)$$

and

$$m_{2n}(a, q) \geq a^n q^{-n^2}, \quad m_{2n+1}(a, q) \geq (a + 1) a^n q^{-n(n+1)}, \quad n \in \mathbb{Z}_+. \quad (55)$$

Proof. It is clear, for instance from (52), that each moment $m_n(a, q)$ is a polynomial in a and q^{-1} with nonnegative integer coefficients. Furthermore, by the very definition (42), $\omega_n(a, q)$ is a polynomial in a of degree n with positive coefficients. From (51) it is seen that

$$m_n(a; q) \leq \frac{\omega_n(a, q)}{(q; q)_{n-1}},$$

and then (53) implies (54).

From (52) one infers that

$$m_{2n+1}(a, q) \geq aq^{-2n}m_{2n-1}(a, q), \quad m_{2n}(a, q) \geq aq^{-2n+1}m_{2n-2}(a, q), \quad \text{for } n \geq 1.$$

Using these inequalities and proceeding by mathematical induction one can verify (55). \square

Appendix. An asymptotic expansion for the basic confluent hypergeometric function and its roots

The purpose of the appendix is to summarize briefly several useful facts about the asymptotic behavior of the basic confluent hypergeometric function which are being referred to in the text of the paper, and to complete them with a few additional observations while making some details more precise. A common interpretation of the studied basic hypergeometric function is within the theory of q -Bessel functions but here we prefer, as mentioned already in the beginning of the paper, to work directly with the function ${}_1\phi_1(0; w; q, z)$. Throughout the appendix we assume that $0 \leq w < 1$ and our focus is on the asymptotic domain $z \rightarrow +\infty$.

In [11], Daalhuis derived a remarkable complete asymptotic expansion of the q -Pochhammer symbol. Let us denote

$$\tilde{q} = e^{4\pi^2/\ln(q)}, \quad \beta(z) = \frac{\pi \ln(z)}{\ln(q)}, \quad (\text{A.1})$$

and

$$A(z) = 2q^{-1/12}\sqrt{z} \exp\left(-\frac{\ln^2(z)}{2\ln(q)} + \frac{\pi^2}{3\ln(q)}\right) |(\tilde{q}e^{-2i\beta(z)}; \tilde{q})_\infty|^2. \quad (\text{A.2})$$

Then

$$(z; q)_\infty = \frac{A(z)}{(q/z; q)_\infty} \sin(\beta(z)) \quad (\text{A.3})$$

for $z > 0$. To facilitate comparison of (A.3) with the original formula in [11] let us note that, for $h > 0$ and β real,

$$\begin{aligned} \exp\left(-\sum_{k=1}^{\infty} \frac{\exp(-hk)}{k \sinh(hk)} \cos(\beta k)\right) &= \left| \exp\left(-\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} e^{(-2hj+i\beta)k}\right) \right|^2 \\ &= |(e^{-2h+i\beta}; e^{-2h})_\infty|^2. \end{aligned}$$

It has also been emphasized in [11] that formula (A.3) has some useful implications for the theta function. Let us point out that this relationship between the q -Pochhammer symbol and the theta function works as well in the opposite direction. As far as notations and basic results related to the theta functions are concerned we refer to [27, Chp. XXI]. The theta function ϑ_1 is known to have the expansion, for β real,

$$\vartheta_1(\beta, q) = 2q^{1/4}(q^2; q^2)_\infty \sin(\beta) \left| (e^{2i\beta} q^2; q^2)_\infty \right|^2 = \sum_{k=0}^{\infty} (-1)^k q^{(2k+1)^2/4} \sin(\beta(2k+1)). \quad (\text{A.4})$$

The so called Jacobi imaginary transformation of ϑ_1 can be written in the form

$$\vartheta_1(\beta, e^{-\pi h}) = \frac{1}{i\sqrt{h}} \exp\left(-\frac{\beta^2}{\pi h}\right) \vartheta_1\left(\frac{i\beta}{h}, e^{-\pi/h}\right), \quad \text{Re } h > 0. \quad (\text{A.5})$$

Let us note that this identity can be derived, for instance, by applying Poisson's summation rule to the series expansion in (A.4). Expressing the theta functions occurring in (A.5) as infinite products one obtains

$$\begin{aligned} & e^{-\pi h/4} \sin(\beta) (e^{-2\pi h}; e^{-2\pi h})_\infty (e^{-2\pi h-2i\beta}; e^{-2\pi h})_\infty (e^{-2\pi h+2i\beta}; e^{-2\pi h})_\infty \\ &= \frac{1}{\sqrt{h}} \exp\left(-\frac{\beta^2}{\pi h}\right) e^{-\pi/(4h)} \sinh\left(\frac{\beta}{h}\right) (e^{-2\pi/h}; e^{-2\pi/h})_\infty \\ & \quad \times (e^{-(2\pi+2\beta)/h}; e^{-2\pi/h})_\infty (e^{-(2\pi-2\beta)/h}; e^{-2\pi/h})_\infty. \end{aligned} \quad (\text{A.6})$$

Differentiating (A.6) with respect to β at $\beta = 0$ one derives a rather neat identity for the q -Pochhammer symbol,

$$(e^{-2\pi h}; e^{-2\pi h})_\infty = \frac{1}{\sqrt{h}} \exp\left(\frac{\pi}{12} \left(h - \frac{1}{h}\right)\right) (e^{-2\pi/h}; e^{-2\pi/h})_\infty, \quad \text{Re } h > 0. \quad (\text{A.7})$$

Furthermore, letting $h = -2\pi/\ln(q)$ and $\beta = \pi \ln(z)/\ln(q)$ in (A.6), and making use of (A.7) one arrives at (A.3).

One may also note that, with increasing z , the term $|\tilde{q} e^{-2i\beta(z)}; \tilde{q})_\infty|$ occurring in (A.2) oscillates (logarithmically) between the extreme values

$$(\tilde{q}; \tilde{q})_\infty = q^{1/24} \sqrt{-\frac{\ln(q)}{2\pi}} \exp\left(-\frac{\pi^2}{6\ln(q)}\right) (q; q)_\infty$$

and

$$(-\tilde{q}; \tilde{q})_\infty = \frac{q^{-1/48}}{\sqrt{2}} \exp\left(-\frac{\pi^2}{6\ln(q)}\right) (q^{1/2}; q)_\infty.$$

By differentiating (A.3) one obtains the asymptotic formula

$$\begin{aligned} \frac{\partial(z; q)_\infty}{\partial z} &= \frac{A(z)}{(q/z; q)_\infty z} \left(\left(-\frac{\beta(z)}{\pi} + \frac{1}{2} + O\left(\frac{1}{z}\right) \right) \sin(\beta(z)) + \frac{\pi}{\ln(q)} \cos(\beta(z)) \right. \\ & \quad \left. + \frac{8\pi}{\ln(q)} \sum_{k=1}^{\infty} \frac{\tilde{q}^k}{|1 - \tilde{q}^k e^{-2i\beta(z)}|^2} \sin^2(\beta(z)) \cos(\beta(z)) \right) \end{aligned} \quad (\text{A.8})$$

as $z \rightarrow +\infty$. Note that the $O(z^{-1})$ term is in fact

$$z(q/z; q)_\infty \frac{\partial}{\partial z} \frac{1}{(q/z; q)_\infty} = -z \frac{\partial}{\partial z} \ln((q/z; q)_\infty) = -\frac{1}{z} \sum_{j=1}^{\infty} \frac{q^j}{1 - q^j z^{-1}}.$$

To deduce from (A.3) and (A.8) some information about the asymptotic behavior of the function ${}_1\phi_1(0; w; q, z)$ for z large one needs the following fundamental relation which has been derived in [18, Prop. 2.1],

$${}_1\phi_1(0; w; q, z) = \frac{(z; q)_\infty}{(w; q)_\infty} {}_1\phi_1(0; z; q, w). \quad (\text{A.9})$$

Theorem A.1. *Let $[x]$ standing for the integer part of $x \in \mathbb{R}$)*

$$K(z) = \left[\frac{1}{2} - \frac{\ln(z)}{\ln(q)} \right]. \quad (\text{A.10})$$

With the notation introduced in (A.1), (A.2), and assuming $0 \leq w < 1$, there exist functions $B(w, z)$ and $C(w, z)$ such that

$$\begin{aligned} {}_1\phi_1(0; w; q, z) &= \frac{B(w, z)}{(w; q)_\infty} \\ &\times \left(A(z) \sin(\beta(z)) + (-1)^{K(z)+1} q^{(K(z)+1)K(z)/2} w^{K(z)+1} \frac{(q^{K(z)+1} z; q)_\infty}{(q; q)_\infty} C(w, z) \right) \end{aligned}$$

and (for a fixed w)

$$B(w, z) = 1 + O(z^{-1}), \quad C(w, z) = 1 + O(z^{-1}) \quad \text{as } z \rightarrow +\infty.$$

Proof. In view of (A.9), we have

$$\begin{aligned} &{}_1\phi_1(0; w; q, z) \quad (\text{A.11}) \\ &= \frac{1}{(w; q)_\infty} \left((z; q)_\infty \sum_{k=0}^{K(z)} \frac{(-1)^k q^{k(k-1)/2} w^k}{(q; q)_k (z; q)_k} + \sum_{k=K(z)+1}^{\infty} \frac{(-1)^k q^{k(k-1)/2} w^k}{(q; q)_k} (q^k z; q)_\infty \right). \end{aligned}$$

Making use of (A.3) one finds that it suffices to put

$$B(w, z) = \frac{1}{(q/z; q)_\infty} \left(1 + \sum_{k=1}^{K(z)} \frac{(-1)^k q^{k(k-1)/2} w^k}{(q; q)_k (z; q)_k} \right), \quad C(w, z) = \frac{\tilde{C}(w, z)}{B(w, z)},$$

where

$$\tilde{C}(w, z) = (q^{K(z)+2}; q)_\infty \left(1 + \sum_{j=1}^{\infty} \frac{(-1)^j q^{(j-1)j/2}}{(q^{K(z)+2}; q)_j} \frac{(q^{K(z)+1} w)^j}{(q^{K(z)+1} z; q)_j} \right).$$

Note that $q^{1/2} \leq q^{K(z)}z < q^{-1/2}$ and so $q^{K(z)} = O(z^{-1})$. Furthermore, for $0 \leq k \leq K(z)$ one has

$$|(z; q)_k| \geq q^{k(k-1)/2} (q^{1/2}; q)_\infty z^k.$$

In fact, this is obviously true for $k = 0$. For $q^{-1/2} \leq z$ and $1 \leq k \leq K(z)$,

$$|(z; q)_k| = q^{k(k-1)/2} z^k (1 - z^{-1})(1 - q^{-1}z^{-1}) \dots (1 - q^{-k+1}z^{-1}) \geq q^{k(k-1)/2} z^k (q^{1/2}; q)_k. \quad (\text{A.12})$$

Hence $B(w, z) = 1 + O(z^{-1})$ as $z \rightarrow +\infty$. The rest of the proof is quite clear. \square

Theorem A.2. *Under the same assumptions as in Theorem A.1,*

$$\begin{aligned} \frac{\partial_1 \phi_1(0; w; q, z)}{\partial z} &= \frac{A(z)}{(w; q)_\infty z} \left(\left(-\frac{\beta(z)}{\pi} + \frac{1}{2} \right) \sin(\beta(z)) + \frac{\pi}{\ln(q)} \cos(\beta(z)) \right. \\ &\quad + \frac{8\pi}{\ln(q)} \sum_{k=1}^{\infty} \frac{\tilde{q}^k}{|1 - \tilde{q}^k e^{-2i\beta(z)}|^2} \sin^2(\beta(z)) \cos(\beta(z)) \\ &\quad \left. + O\left(\frac{\ln(z)}{z}\right) \right) \end{aligned}$$

as $z \rightarrow +\infty$.

Proof. Using the same notation as in the proof of Theorem A.1 one again starts from equation (A.11). Note that

$$q^{(K(z)+1)K(z)/2} < \exp\left(\frac{\ln(q)}{2} \left(\frac{\ln^2(z)}{\ln^2(q)} - \frac{1}{4}\right)\right) = q^{-1/8} \exp\left(\frac{\ln^2(z)}{2 \ln(q)}\right).$$

Furthermore, for $k > K(z)$ we have $0 < (q^k z; q)_\infty < 1$ and

$$\left| \frac{\partial}{\partial z} \ln((q^k z; q)_\infty) \right| = \sum_{j=k}^{\infty} \frac{q^j}{1 - q^j z} \leq \frac{q^{K(z)+1}}{(1 - q^{1/2})(1 - q)} < \frac{q^{1/2}}{(1 - q^{1/2})(1 - q)}.$$

For $1 \leq k \leq K(z)$, we again have (A.12) and also

$$\left| \frac{\partial}{\partial z} \ln((z; q)_k) \right| = \frac{1}{z} \sum_{j=0}^{k-1} \frac{q^j z}{q^j z - 1} \leq \frac{k}{z(1 - q^{1/2})}.$$

Consequently,

$$\frac{\partial_1 \phi_1(0; w; q, z)}{\partial z} = \frac{1}{(w; q)_\infty} \frac{\partial(z; q)_\infty}{\partial z} (1 + O(z^{-1})) + \frac{(z; q)_\infty}{(w; q)_\infty} O(z^{-2}) + O\left(\exp\left(\frac{\ln^2(z)}{2 \ln(q)}\right)\right).$$

Recalling (A.3) and (A.8) we obtain the sought formula. \square

Zeros of the q -Bessel functions have been studied in a number of papers. For the equation ${}_1\phi_1(0; w; q, z) = 0$ in the complex variable z (with $0 \leq w < 1$ being fixed) these results mean that the roots are all positive and simple [16]. Ordering the roots increasingly, $\zeta_0 < \zeta_1 < \zeta_2 < \dots$, the leading asymptotic term of ζ_m for large m was derived in [1, 3]. In more detail, Annaby and Mansour showed in [3, Thm. 2.2] that

$$\zeta_m = q^{-m} + O(1) \quad \text{as } m \rightarrow \infty. \quad (\text{A.13})$$

On the basis of Theorem A.1 we can augment this result by showing that $q^m \zeta_m$ approaches the value 1 much faster than one might guess from (A.13).

Proposition A.3. *Denote by $0 < \zeta_0 < \zeta_1 < \zeta_2 < \dots$ the increasingly ordered roots of the equation ${}_1\phi_1(0; w; q, z) = 0$ in the variable z . Then*

$$\zeta_m = q^{-m} - \frac{w^{m+1} q^{m^2}}{(q; q)_\infty^2} (1 + O(q^m)) \quad \text{as } m \rightarrow \infty.$$

Proof. Recalling (A.10), one can see from (A.13) that $K(z) = m$ on quite a large neighborhood of ζ_m . More precisely, if

$$q^{-m}(q^{1/2} - 1) \leq z - q^{-m} < q^{-m}(q^{-1/2} - 1)$$

then $K(z) = m$. Bearing in mind equation (A.13) we write $\zeta_m = q^{-m} + \epsilon_m$ while assuming ϵ_m to be bounded. According to Theorem A.1 we have to solve the equation

$$\sin\left(\frac{\pi \ln(1 + q^m \epsilon_m)}{\ln(q)}\right) = \frac{q^{(m+1)m/2} w^{m+1}}{A(q^{-m} + \epsilon_m)} \frac{(q + q^{m+1} \epsilon_m; q)_\infty}{(q; q)_\infty} C(w, q^{-m} + \epsilon_m).$$

Since $A(q^{-m} + \epsilon_m) = A(q^{-m}) (1 + O(q^m))$ and $C(w, q^{-m} + \epsilon_m) = 1 + O(q^m)$, one has

$$\frac{\pi q^m \epsilon_m}{\ln(q)} = \frac{q^{(m+1)m/2} w^{m+1}}{A(q^{-m})} (1 + O(q^m)).$$

Recalling (A.7) one finds that

$$A(q^{-m}) = -\frac{\ln(q)}{\pi} q^{-(m+1)m/2} (q; q)_\infty^2.$$

The result readily follows. □

Corollary A.4. *Under the same assumptions as in Proposition A.3,*

$$\frac{\partial {}_1\phi_1(0; w; q, \zeta_k)}{\partial z} = (-1)^{k+1} \frac{(q; q)_\infty^2}{(w; q)_\infty} q^{-k(k-1)/2} (1 + O(q^k)) \quad \text{as } k \rightarrow \infty.$$

Proof. The formula follows immediately from Theorem A.2 and Proposition A.3 if taking into account (A.7). □

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